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Light-cone lattices and the exact solution of chiral fermion and sigma models

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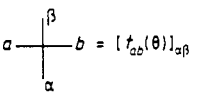
Abstract. A rich set of integrable two-dimensional quantum field theories are obtained from integrable lattice vertex models with q states per bound ($q \geqslant 2$) in the scaling limit by a generalisation of the light-cone lattice approach. Chiral fermion models with any simple Lie group of symmetry arise in this way (for finite q) as well as bosonic models like the principal chiral model (for $q = \infty$). The Hamiltonian, momentum and colour-conserved currents are constructed on the lattice and the bare equations of motion are derived. The renormalised mass spectrum is given explicitly for the set of models considered here. All these integrable vertex models yield conformal invariant theories if one takes the scaling limit in an appropriate different way. It is argued that the values one obtains for the central charges are the same as those provided by the Sugawara construction (in the continuum) for all simple Lie algebras.

1. Introduction and summary

The construction of exact solutions of 2D integrable statistical models has made impressive progress in recent years [1]. Eigenvalues and eigenvectors of a rich set of theories have been constructed by means of the Bethe ansatz (BA) and its generalisations.

Usually, the physics of a statistical model at criticality (infinite correlation length) can be described by a continuum quantum field theory. Therefore, integrable lattice models in their scaling limit seem to be very appropriate starting points of a programme for the building of exactly solvable quantum field theories (QFT).

As a first step in this programme, the massive Thirring model (MTM) fields are constructed from an exactly solvable lattice theory, the six-vertex model, in [2]. The light-cone lattice approach used there is actually appropriate for systematically building various integrable fermionic and bosonic QFT from vertex models (figure 1), taking the scaling limit in the precise way illustrated in the present paper.



$= [t_{ab}(\theta)]_{\alpha\beta}$

Figure 1. Statistical weight associated with the depicted configuration, $1 \leqslant \alpha, \beta, a, b, \leqslant q$.

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Lattice vertex models are usually treated using the row-to-row transfer matrix (figure 2). However, a simple relation exists between the light-cone (diagonal-to-diagonal) transfer matrices and the inhomogeneous row-to-row transfer matrices. By inhomogeneous we mean when the spectral argument θ of the vertex weights depends on the column of the lattice as

$$\theta \to \tfrac{1}{2}(\theta_0 - \theta_n) \qquad 1 \leqslant n \leqslant N.$$

Setting $\theta_n = \theta\ (-\theta)$ for odd (even) columns leads to a light-cone transfer matrix when one sets $\theta_0 = \theta$ or $\theta_0 = -\theta$, as is easily seen in figures 2-4. (An analytic proof is given in § 2.) More precisely

$$\begin{aligned} &\tau(\theta; \{\theta_n = (-)^{n+1}\theta\}) = U_{\mathrm{L}}(\theta) \\ &\tau(-\theta; \{\theta_n = (-)^{n+1}\theta\}) = U_{\mathrm{R}}(\theta)^{+}. \end{aligned} \tag{1.1}$$

Here U_{L} and U_{R} respectively produce a right or left unit translation in the diagonal direction over the lattice. U_{R} and U_{L} are depicted in figure 4.

The lattice Hamiltonian and momentum are defined as

$$H \pm P = (2\mathrm{i}/a) \log U_{\mathrm{R,L}}(\theta) \tag{1.2}$$

where a is the lattice spacing. All known integrable vertex models possess gapless regimes where one can define a scaling limit $a \to 0$ such that a relativistic theory emerges. That is, we choose $\theta = \theta(a)$ such that a (non-empty) set of massive states remains in the spectrum of H and P in the $a \to 0$ limit. We want to stress that several inequivalent QFT may be obtained from a single lattice model by choosing different functions $\theta = \theta(a)$. Specific examples are discussed in § 2. In table 1 we summarise the mass spectrum μm_l for the fundamental models symmetric under all the different simple

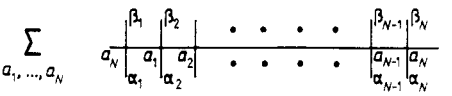


Figure 2. The row-to-row transfer matrix, equation (2.1).

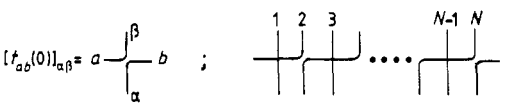


Figure 3. Graphical representation of the regularity condition equation (2.5) and the transfer matrix $\tau(-\theta, \theta)$.

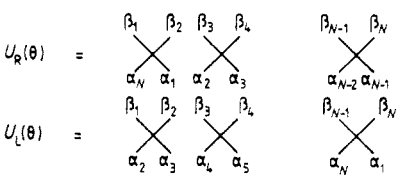
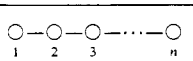
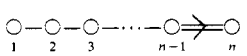
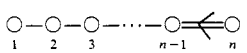
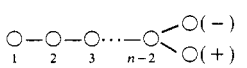
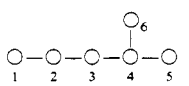
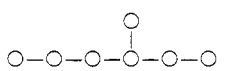
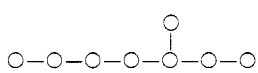
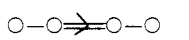


Figure 4. Diagonal-to-diagonal transfer matrices (N is even).

Table 1.

Lie algebra	Dynkin diagram	κ	m_k
A_n		$2\pi/n+1$	$\sin(\pi k/n+1), 1\leqslant k\leqslant n$
B_n		$\pi/2n-1$	$\sin(\pi k/2n-1), 1\leqslant k\leqslant n-1$; $m_n=\frac{1}{2}$
C_n		$\pi/n+1$	$\sin[\pi k/2(n+1)], 1\leqslant k\leqslant n$
D_n		$\pi/n-1$	$\sin[\pi k/2(n-1)], 1\leqslant k\leqslant n-2$; $m_\pm=\frac{1}{2}$
E_6		$\pi/6$	$m_1=m_5=m_6/2=\sqrt{3}/2$; $m_2=m_4=(3+\sqrt{3})/2$; $m_3=(3+\sqrt{3})/\sqrt{2}$
E_7		$\pi/9$	†
E_8		$\pi/15$	†
F_4		$\pi/9$	†
G_2		$\pi/6$	†

† These values can be (very laboriously) extracted from [4, 6].

Lie algebras. The mass unit μ is related in the scaling limit to the lattice spacing a and the parameter θ via

$$\mu = (1/\pi a)\exp(-\kappa\theta) \tag{1.3}$$

where $a\to 0$ and $\theta\to\infty$ such that μ is fixed (the values of κ are listed in table 1). In addition to the mass spectrum, the exact S matrix can be derived from the Bethe ansatz equations which determine the eigenstates of $\tau(\theta_0;\{\theta\})$ (which are also eigenstates of U_R and U_L).

All rational R matrices (i.e., matrices $R(\theta)$ depending rationally on θ) lead to gapless vertex models (see equation (2.3) or figure 1), and therefore a relativistic QFT can be constructed in the scaling limit (1.3). Rational R matrices invariant under any simple Lie group G are known [3, 4]. For large values of the spectral parameter θ they behave as

$$R(\theta) = P[1+(1/i\theta)(\Pi+\lambda)+O(1/\theta^2)] \tag{1.4}$$

where λ is a numerical constant, P is the exchange operator, $P_{ab,cd}=\delta_{ad}\delta_{bc}$, and

$$\Pi = 2\sum_{\alpha=1}^{\dim G} T^\alpha\otimes T^\alpha \tag{1.5}$$

where the T^α are the generators of G in the fundamental representation, normalised by $\operatorname{Tr} T^\alpha T^\beta = l_0\delta^{\alpha\beta}$, $l_0>0$. It must be realised that (1.4) is just the semiclassical expansion for the R matrix. One can then introduce the lattice operator (see § 2)

$$T_n^\alpha = \mathbb{1}\otimes\ldots\overbrace{\otimes\, T^\alpha\otimes}^{n\text{th site}}\ldots\otimes\mathbb{1}. \tag{1.6}$$

As is shown in § 2, T_n^α obey local equations of motion on the lattice:

$$U_R T_{2n-2}^\alpha U_R^\dagger = U_L T_{2n}^\alpha U_L^\dagger$$
$$= T_{2n}^\alpha + (2i/\theta) f^{\alpha\beta\gamma} T_{2n-1}^\beta T_{2n}^\gamma + O(1/\theta^2) \tag{1.7a}$$
$$U_R T_{2n-1}^\alpha U_R^\dagger = U_L T_{2n+1}^\alpha U_L^\dagger$$
$$= T_{2n-1}^\alpha - (2i/\theta) f^{\alpha\beta\gamma} T_{2n-1}^\beta T_{2n}^\gamma + O(1/\theta^2). \tag{1.7b}$$

A closed form for the terms of $O(1/\theta^2)$ is known for $G = SU(n)$ (equation (2.24)). A bare scaling limit, $\theta \to \infty$ and then $a \to 0$, can be defined such that (1.6) yields the continuous zero-divergence and zero-curvature conditions on the vector current

$$J_R^\alpha(x) = \frac{1}{ga\theta} T_{2n}^\alpha \qquad J_L^\alpha(x) = \frac{1}{ga\theta} T_{2n-1}^\alpha \qquad x \simeq na. \tag{1.8}$$

Therefore we have a lattice version of the $\dot{G}$ algebra currents $J_\mu^\alpha(x)$ in the context of an integrable discretisation of field-theoretic models.

Gapless lattice models exhibit conformal invariant behaviour for distances much larger than the lattice spacing a. Therefore, conformal field theories arise in the limit

$$a \to 0 \qquad \frac{d\theta}{da} = 0. \tag{1.9}$$

Notice that this scaling limit differs from (1.3). The finite-size corrections to the free energy and excitation energies yield the conformal properties of the model, that is the central charge c and the conformal dimensions $\Delta, \bar{\Delta}$. The value of c is known exactly for all fundamental models associated with simply laced Lie algebras [5, 6] and for the spin-S SU(2) symmetric model [7]. These results indicate the general formula for the central charge

$$c = x \dim G/(x + \tilde{h}). \tag{1.10}$$

Here dim G is the dimension of the Lie algebra, $\tilde{h}$ its dual Coxeter number† and x stands for the number of fundamental R matrices fused to produce the model. We have $x = 1$ for fundamental models and $x = 2S$ for the spin-S SU(2) symmetric model. The Sugawara construction of the Virasoro algebra given in [8] also satisfies (1.10). We believe that the gapless integrable theories associated with a Lie algebra $\dot{G}$ provide, through their long-distance behaviour, an alternative construction of the conformal algebra. It is interesting to note that models like the six-vertex model, where $c = 1$, explicitly provide an infinite number of primary fields in the long-distance regime.

Conformal field theories appear, therefore, in the limit (1.9) of lattice integrable models. These integrable models clearly have a much richer structure than the continuum field theories since, for example, they yield massive QFT in the limit (1.3).

In § 3, we review the coordinate Bethe ansatz (CBA) for a general class of two-dimensional chiral fermionic models (CFM) defined by the Lagrangian

$$\mathcal{L} = i\bar{\psi}\partial\psi - \tfrac{1}{4}(\bar{\psi}_a \gamma_\mu \psi_c) V_{ab,cd} (\bar{\psi}_b \gamma^\mu \psi_d) \tag{1.11}$$

where ψ_a is a Dirac field. The CBA solution shows that this model describes massless free particles bearing non-zero chirality plus massive excitations. Actually, the CBA

† It is given by $\tilde{h} = C_2/\|\alpha\|$, where C_2 is the quadratic Casimir operator in the adjoint representation and $\|\alpha\|$ the length of the longest root of the algebra.

only works when the interaction matrix V yields a two-body S matrix (3.12) obeying the YB equation (3.19). Then the massive sector of (1.11) coincides with the scaling limit (1.2), (1.3) of the lattice model built from this S matrix as vertex weights (equation (3.21)). Together with the lattice current construction (1.6)–(1.8), this provides an explicit connection between the CBA approach in the continuum and the light-cone transfer matrix approach on the lattice.

We also check in § 3 that the one-loop perturbative β function for the CFM is correctly reproduced by the BA for all fundamental models associated with simple Lie algebras. This is a highly non-trivial check in view of the completely different mathematical structure of the continuum Feynman diagrams and the lattice or CBA. Actually, the CBA approach has serious drawbacks and may sometimes lead to wrong results. This is illustrated in § 3 for the massless Thirring model and in § 4 for the multiflavour generalisation of (1.11), i.e.

$$\mathscr{L} = \mathrm{i}\bar{\psi}_r \partial \psi_r - \tfrac{1}{2} g(\bar{\psi}_r \gamma_\mu T^\alpha \psi_r)(\bar{\psi}_{r'} \phi^\mu T^\alpha \psi_{r'}) \tag{1.12}$$

where each $\psi_r (1 \leqslant r \leqslant N_f)$ transforms under an irreducible representation ρ of the group G, and T_α are the generators of G in that representation. In (1.12), we used

$$V_{ab,cd} = g \Pi_{ab,cd}$$

where Π is given by (1.5).

In § 4 we also briefly analyse the connection between the $N_f \to \infty$ limit of the multiflavour chiral Gross–Neveu (MCGN) model and the SU(n) principal chiral σ model. For $n = 2$, the latter can also be (partially) obtained in the $S \to \infty$ limit of the spin-S SU(2) light-cone vertex model.

In summary, we shall show in this paper how to generate integrable (massive) QFT and conformal QFT out of integrable vertex models by performing appropriate scaling limits. The scope of the light-cone lattice approach for integrable QFT seems to cover a very general set of models including all previously known examples.

2. Light-cone versus row-to-row transfer matrices and field-theoretic models

The row-to-row transfer matrix is one of the fundamental objects for an integrable vertex model. Let us consider an inhomogeneous vertex model where both horizontal and vertical links take values in the same q-dimensional vector space (see figure 1). The row-to-row transfer matrix is

$$\tau(\theta_0; \{\theta_i\}) = \sum_{a_1 \ldots a_N = 1}^{q} t_{a_1 a_2}\left(\frac{\theta_0 - \theta_1}{2}\right) t_{a_2 a_3}\left(\frac{\theta_0 - \theta_2}{2}\right) \ldots t_{a_N a_1}\left(\frac{\theta_0 - \theta_N}{2}\right). \tag{2.1}$$

It is graphically depicted in figure 2. θ_0 is the spectral parameter and θ_i $(1 \leqslant i \leqslant N$, with N an even number) are the inhomogeneity parameters.

The vertex model is said to be integrable if the weights $t_{ab}(\theta)$ obey the Yang–Baxter (YB) algebra

$$R(\theta - \theta')[t(\theta) \otimes t(\theta')] = [t(\theta') \otimes t(\theta)] R(\theta - \theta') \tag{2.2}$$

where

$$R^{\beta b}_{a\alpha}(\theta) = [t_{ab}(\theta)]_{\alpha\beta}. \tag{2.3}$$

It follows from (2.1) and (2.2) that

$$[\tau(\theta_0; \{\theta_i\}), \tau(\theta_0'; \{\theta_i\})] = 0 \qquad \forall \theta_0, \theta_0'. \tag{2.4}$$

We choose here a regular solution of the YB equations (2.2) such that

$$[t_{ab}(0)]_{\alpha\beta} = \delta_{a\beta}\delta_{b\alpha}. \tag{2.5}$$

Let us now make the following choice of inhomogeneities:

$$\theta_k = (-1)^{k+1}\theta \qquad k = 1, 2, \ldots, N. \tag{2.6}$$

In this case the sum over $a_j (1 \leqslant j \leqslant q)$ in (2.1) can be trivially performed, using (2.5), whenever $\theta_0 = \theta$ or $\theta_0 = -\theta$:

$$\begin{aligned}\tau(\theta, \theta)_{\boldsymbol{\alpha},\boldsymbol{\beta}} &\equiv \tau(\theta; \{\theta_k = (-1)^{k+1}\theta\})_{\alpha_1 \ldots \alpha_N, \beta_1 \ldots \beta_N} \\ &= [t_{\alpha_1\beta_3}(\theta)]_{\alpha_2\beta_2}[t_{\alpha_3\beta_5}(\theta)]_{\alpha_4\beta_4} \ldots [t_{\alpha_{2N-1}\beta_1}(\theta)]_{\alpha_N\beta_N}.\end{aligned} \tag{2.7}$$

Using (2.3), this becomes

$$\tau(\theta, \theta)_{\boldsymbol{\alpha},\boldsymbol{\beta}} = R^{\beta_2\beta_3}_{\alpha_1\alpha_2}(\theta) \ldots R^{\beta_N\beta_1}_{\alpha_N\alpha_1}(\theta). \tag{2.8}$$

But this is precisely a diagonal-to-diagonal, or light-cone, transfer matrix. In the notation of [2],

$$\tau(\theta, \theta) = U_+(\theta) V^+ = U_L(\theta). \tag{2.9}$$

Similarly, if we consider

$$\tau(-\theta, \theta) \equiv \tau(-\theta; \{\theta_k = (-1)^{k+1}\theta\}) \tag{2.10}$$

we find

$$\tau(-\theta, \theta) = V^+ U_+(-\theta) = [U_+(\theta) V]^+ = U_R(\theta)^+ \tag{2.11}$$

since $R(\theta)^+ = R(-\theta) = R(\theta)^{-1}$ (a property which can always be achieved by proper normalisations).

Therefore the light-cone transfer matrices $U_{R,L}(\theta)$ are particular cases of an inhomogeneous row-to-row transfer matrix with alternating inhomogeneities given by (2.6). The commutativity property (2.4) yields therefore

$$\begin{aligned}&[\tau(\theta_0, \theta), U_L(\theta)] = 0 = [\tau(\theta_0, \theta), U_R(\theta)] \\ &[U_R(\theta), U_L(\theta)] = 0\end{aligned} \tag{2.12}$$

where $\tau(\theta_0, \theta) \equiv \tau(\theta_0; \{\theta_k = (-1)^{k+1}\theta\})$. One can consider the infinite sequence of commuting operators

$$c_k^{\pm} = \frac{1}{k!} \frac{\partial^k \tau(\theta_0, \theta)}{\partial \theta_0^k}\bigg|_{\theta_0 = \pm\theta} \qquad k = 1, 2, \ldots. \tag{2.13}$$

They all commute with $U_L(\theta)$ and $U_R(\theta)$:

$$[c_k^{\pm}, U_L(\theta)] = [(c_k^{\pm}, U_R(\theta)] = 0. \tag{2.14}$$

As explained in [2], the operators $U_R(\theta)$ and $U_L(\theta)$ are precisely the light-cone evolution operators in a discretised Minkowski spacetime for the MTM. We shall generalise this interpretation to a large class of QFT. We define the Hamiltonian and momentum operators for the lattice QFT as

$$H_{R,L} = \tfrac{1}{2}(H \pm P) = (\mathrm{i}/a) \log U_{R,L}(\theta) \tag{2.15}$$

where a is the lattice spacing. The continuum limit is obtained by letting $\theta = \theta(a)$ in such a way that the physical mass gap is finite. Let us consider the spectra of $U_R(\theta)$ and $U_L(\theta)$ in order to find $\theta(a)$ and the mass spectrum.

Since U_R and U_L are expressed in terms of the row-to-row transfer matrix (equations (2.9) and (2.11)), it is enough to analyse the spectrum of $\tau(\theta_0; \{\theta_k\})$. The eigenvectors and eigenvalues can be exactly computed by the BA and its nested generalisations [1, 9]. The ground-state eigenvector just corresponds (in the scaling limit) to the vacuum state of the QFT. The particle states follow from the lowest-lying excitations. Since there is a factor a^{-1} in the Hamiltonian (see equation (2.15)), only gapless vertex models may yield finite energy states in the scaling limit. Therefore, in order to calculate the energy and momentum in the scaling limit, it is enough to know the eigenvalues of $\tau(\theta_0; \{\theta_k\})$ close to the bottom of the spectrum.

Let us take the family of fundamental vertex models associated with simple Lie algebras [3, 4] (other models will be considered later on). The low-lying excitations in these models are associated with holes with large rapidity ϕ. Moreover, the large-ϕ behaviour of the eigenvalues is independent of the inhomogeneity parameters θ_j $(1 \leqslant j \leqslant N)$; its explicit form was derived in [6]:

$$f_l(\phi) \equiv -\lim_{N\to\infty} \log[\Lambda_{\max}(\theta_0; \{\theta_k\})^{-1} \Lambda_l(\phi, \theta_0; \{\theta_k\})]$$

$$\underset{\phi+\theta_0\to\pm\infty}{\simeq} \frac{\mathrm{i}m_l}{\pi} \exp[\mp\kappa(\phi+\theta_0)] + \mathrm{O}\{\exp[\mp 2\kappa(\phi+\theta_0)]\}. \tag{2.16}$$

Here $\Lambda_l(\phi, \theta_0; \{\theta_k\})$ is the contribution of a hole in the lth branch $(1 \leqslant l \leqslant \text{rank G})$ to the BA eigenvalue of $\tau(\theta_0; \{\theta_k\})$. The dimensionless parameters κ and m_l are given in table 1 for all these models associated with simple Lie algebras in their fundamental representations.

Combining (2.16) with (2.9), (2.11) and (2.15) yields the energy–momentum dispersion relations (with $\theta \to \infty$ and an appropriate choice of the logarithm branch in (2.15))

$$e_l(\phi) = \left(\frac{\exp(-\kappa\theta)}{\pi a}\right) m_l \cosh(\kappa\phi) + \mathrm{O}[\exp(-2\kappa\theta)] \tag{2.17a}$$

$$p_l(\phi) = \left(\frac{\exp(-\kappa\theta)}{\pi a}\right) m_l \sinh(\kappa\phi) + \mathrm{O}[\exp(-2\kappa\theta)]. \tag{2.17b}$$

We then define the scaling limit according to

$$a \to 0 \qquad \theta \to \infty \qquad \mu \equiv \exp(-\kappa\theta)/\pi a \text{ fixed}. \tag{2.18}$$

μ is the renormalised, or physical, mass scale and the mass spectrum of these integrable QFT is given by

$$M_l = \mu m_l. \tag{2.19}$$

From the relativistic form of the dispersion law (2.17), we also recognise in $\kappa\phi$ the physical rapidity of the particles.

This is a very general way of constructing integrable QFT. The operators H and P given by (2.15) are well defined on the lattice, as are all the higher conserved charges (2.13). In the continuum limit $a \to 0$, they provide the energy and momentum of a relativistic invariant QFT, as long as the spectrum of the original vertex model is gapless. This is usually the case for statistical weights $t_{ab}(\theta)$ which are rational or trigonometric functions of the spectral parameter θ. In addition to the particle spectrum, the S matrix is exactly calculable from the BA equation by standard methods [1].

As was the case for the MTM, the evolution operators U_R and U_L of the light-cone lattice are much simpler than the Hamiltonian and momentum. This fact was exploited in [2] to obtain a lattice field equation for the bare fundamental fields of the MTM regularised on the lattice. An analogous local construction of lattice field operators would be very interesting to obtain in the general case. We present here a lattice construction for the currents of a large class of integrable models that will be specified below. Let us first give an example of the construction in a particular model, in which the R matrix $R(\theta)$ (symmetric under the action of SU(n) in the fundamental representation, so that $q = n$ in this case) is:

$$R(\theta) = (1 + i\theta P)/(1 + i\theta). \tag{2.20}$$

Here $P_{ab,cd} = \delta_{ad}\delta_{bc}$ ($1 \leqslant a, b, c, d \leqslant n$) can be written in terms of the SU(n) Hermitian generators T^α as

$$P = \frac{1}{n} + \pi \qquad \pi \equiv 2\sum_{\alpha=1}^{n^2-1} T^\alpha \otimes T^\alpha. \tag{2.21}$$

The T^α are such that

$$[T^\alpha, T^\beta] = i f^{\alpha\beta\gamma} T^\gamma \qquad \operatorname{Tr} T^\alpha T^\beta = \tfrac{1}{2}\delta^{\alpha\beta}$$

where $f^{\alpha\beta\gamma}$ are totally antisymmetric SU(n) structure constants.

We define the SU(n) currents on the lattice by attaching a T^α generator to each link of a unit horizontal slice of the diagonal lattice (see figure 5)

$$T_n^\alpha = \mathbb{1} \otimes \ldots \overbrace{\otimes\, T^\alpha \otimes}^{n\text{th link}} \ldots \otimes \mathbb{1} \qquad 1 \leqslant n \leqslant N. \tag{2.22}$$

We must apply $U_{L,R}(\theta)$ and $U_{L,R}(\theta)^+$ to T_n^α from the left and right respectively, in order to derive the unit space and time evolutions of T_n^α on the lattice. These calculations reduce to the following local algebra (compare figures 4 and 5):

$$R(\theta)(T^\alpha \otimes \mathbb{1})R(\theta)^+ = \frac{1}{1+\theta^2}[\theta^2(\mathbb{1} \otimes T^\alpha) + T^\alpha \otimes \mathbb{1} + 2i\theta f^{\alpha\beta\gamma} T^\beta \otimes T^\gamma] \tag{2.23a}$$

$$R(\theta)(\mathbb{1} \otimes T^\alpha)R(\theta)^+ = \frac{1}{1+\theta^2}[\theta^2(T^\alpha \otimes \mathbb{1}) + \mathbb{1} \otimes T^\alpha - 2i\theta f^{\alpha\beta\gamma} T^\beta \otimes T^\gamma]. \tag{2.23b}$$

Therefore

$$\begin{aligned} U_R T_{2n-2}^\alpha U_R^+ &= U_i T_{2n}^\alpha U_L^+ \\ &= \frac{1}{1+\theta^2}(\theta^2 T_{2n}^\alpha + T_{2n-1}^\alpha + 2i\theta f^{\alpha\beta\gamma} T_{2N-1}^\beta T_{2N}^\gamma) \end{aligned} \tag{2.24a}$$

$$\begin{aligned} U_R T_{2n-1}^\alpha U_R^+ &= U_L T_{2n+1} U_L^+ \\ &= \frac{1}{1+\theta^2}(\theta^2 T_{2n-1}^\alpha + T_{2n}^\alpha - 2i\theta f^{\alpha\beta\gamma} T_{2n-1}^\beta T_{2n}^\gamma). \end{aligned} \tag{2.24b}$$

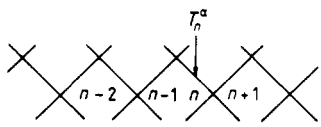


Figure 5. The lattice current T_n^α associated with the nth link.

These are the bare equations for the currents on the lattice. In order to obtain the corresponding continuous equations, we must take a (bare) scaling limit. As was already the case for the MTM, bare and renormalised scaling limits are different.

The bare scaling limit consists here of two steps.

(i) Define $T_n^\alpha = g\theta t_n^\alpha$ and let $\theta \to \infty$ with the parameter g fixed (g will be the bare coupling constant). Equations (2.23) and (2.24) yield in this limit

$$\begin{aligned} U_R t_{2n-2}^\alpha U_R^\dagger &= U_L t_{2n}^\alpha U_L^\dagger = t_{2n}^\alpha + 2ig f^{\alpha\beta\gamma} t_{2n-1}^\beta t_{2n}^\gamma \\ U_R t_{2n-1}^\alpha U_R^\dagger &= U_L t_{2n+1}^\alpha U_L^\dagger = t_{2n-1}^\alpha - 2ig f^{\alpha\beta\gamma} t_{2n-1}^\beta t_{2n}^\gamma . \end{aligned} \tag{2.25}$$

(ii) Let the lattice spacing a tend to zero. Then we can set

$$\begin{aligned} &U_A \simeq 1 - iaH_A \qquad A = R, L \\ &t_{2n}^\alpha = aJ_R^\alpha(x) \qquad t_{2n-1}^\alpha = aJ_L^\alpha(x) \qquad x \simeq na. \end{aligned} \tag{2.26}$$

From (2.25) and (2.26), we find for $a \to 0$

$$\begin{aligned} &\partial_\mu J^{\mu\alpha} = \partial_0 J_0^\alpha - \partial_1 J_1^\alpha = 0 \\ &\partial_\mu J_\nu^{\ \alpha} - \partial_\nu J_\mu^{\ \alpha} + igf^{\alpha\beta\gamma}[J_\mu^{\ \beta}, J_\nu^{\ \gamma}] = 0 \end{aligned} \tag{2.27}$$

where $J_0 = \frac{1}{2}(J_R + J_L)$ and $J_1 = \frac{1}{2}(J_R - J_L)$. Therefore the lattice operators T_n^α provide an integrable discretisation of the zero-divergence and zero-curvature equations (2.27). These equations characterise the currents in the SU(n) CGN model. In addition, H and P provided by (2.15), using the R matrix (2.20), are the discretised (integrable) versions of the CGN Hamiltonian and momentum in the zero-chirality sector (see § 3).

Let us now generalise the current construction for all rational R matrices admitting the large θ (semiclassical) expansion [10]

$$R(\theta) = P(1 + (1/i\theta)(\Pi + \lambda) + O(1/\theta^2)) \tag{2.28}$$

where λ is a numerical constant, $\Pi = \Sigma_{\alpha\beta} K^{\alpha\beta} t_\alpha \otimes t_\beta$, with t_α being the generators of the simple Lie algebra $\dot{G}$ in some representation ρ and $K^{\alpha\beta}$ being proportional to the inverse of the Killing form. Solutions of the YB equations of this form exist for all representations ρ when $\dot{G} = A_n$ for some ρ, including the fundamental representation, for all the other simple Lie algebras. As in the SU(n) case, we define the lattice current operator by (2.22). Relations analogous to (2.23) and (2.24) hold in the general case (2.28), but the details of the terms of order θ^{-2} depend on the specific Lie algebra $\dot{G}$ and representation ρ chosen. However, in the bare scaling limit, $\theta \to \infty$, only the leading terms precisely provided by (2.28) survive. Therefore (2.25)-(2.27) hold in the general case where $f^\alpha{}_{\beta\gamma}$ now stand for the structure constants of G. We achieve in this way an integrable discretisation for the currents in the integrable models associated with a rational R matrix symmetric under $\dot{G}$ in the representation ρ. For finite-dimensional ρ, this corresponds in the continuum to the zero-chirality sector of the chiral invariant model [11]

$$\mathcal{L} = \bar{\psi}\partial\psi - \tfrac{1}{4} g K^{\alpha\beta} (\bar{\psi}\gamma_\mu t_\alpha \psi)(\bar{\psi}\gamma^\mu t_\beta \psi) \tag{2.29}$$

where we can identify

$$J_\mu^{\ \alpha} = \bar{\psi}\gamma_\mu t^\alpha \psi \tag{2.30}$$

although no lattice version of ψ is available at present (except for the MTM [2]). Moreover, it is shown in § 3 (equations (3.18)-(3.23)) that the Hamiltonian and momentum derived from (2.29) by CBA coincide with the lattice operators (2.15) in the zero-chirality massive sector. It must be noted that (2.29) can be rewritten as in (1.11) with $V = g\Pi$.

For infinite-dimensional representations ρ the lattice models can be related to continuum bosonic models like the PC σ model (see below).

The spin-S ($S>\frac{1}{2}$) generalisation of the six-vertex model (with $2S+1$ states per link rather than just 2) provides a family of gapless vertex models associated with the Lie algebra $\dot{G}=A_1$ [12]. These models are characterised, besides the spectral parameter θ, by an anisotropy parameter $\gamma(0\leqslant\gamma<\pi)$, with $\gamma\to 0$ corresponding to the isotropic SU(2)-invariant limit. It is convenient to represent π/γ as a continued fraction

$$p_0=\frac{\pi}{\gamma}=b_0+\cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{b_3+\dots}}} \tag{2.31}$$

where $b_0, b_1, b_2, \dots$ are positive integers or zero. One also introduces the numbers p_i, q_i, y_i and n_i according to

$$\begin{aligned}
&p_1=1 \qquad p_{i+1}=p_{i-1}-b_{i-1}p_i \qquad b_i\equiv[p_i/p_{i+1}] \qquad i\geqslant 1\\
&q_0=0 \qquad q_1=b_0 \qquad q_{i+1}=q_i+b_1 \qquad i\geqslant 0\\
&y_{-1}=0 \qquad y_0=1 \qquad y_{i+1}=y_{i-1}+b_iy_i \qquad i\geqslant 0\\
&n_j=y_{i-1}+(j-q_i)y_i \qquad q_i\leqslant j\leqslant q_{i+1} \qquad i\geqslant 0.
\end{aligned} \tag{2.32}$$

The analysis of [12] holds when there exist numbers r and σ such that $q_r\leqslant\sigma\leqslant q_{r+1}$, $2S+1=n_\sigma$. Then one finds a family of low-energy excitations associated with each type of strong configuration (with string length n_i) present in the ground state [12]. There exists in each family a kink state with mass

$$M_j=(m_j/a)\exp(-\pi\theta/2p_{j-1}) \tag{2.33}$$

where the number m_j does not depend on θ and the index j, which labels the families, takes all even values from 0 up to r (r even) or $r+1$ (r odd). In addition, one finds breather states with masses

$$M_j^{(k)}=2M_j\sin\left((k-q_{j-1}+1)\frac{\pi p_j}{2p_j-1}\right) \qquad q_{j-1}\leqslant k<q_j. \tag{2.34}$$

Therefore one can take the scaling limit in $[(r+1)/2]+1$ different ways. That is, one can keep fixed any one of the kink masses (2.33), say M_l, as $a\to 0$ and $\theta\to\infty$ (cf equation (2.18)). The families with mass scale smaller or larger than M_l become, respectively, massless or infinitely heavy in the scaling limit. They decouple in both cases from the particles of the lth family. The physical S matrix can be found in [12].

The field-theoretic models discussed up to here correspond to finite values of q, namely a finite-dimensional vector space for each link in the light-cone lattice. This is clearly appropriate for fermionic fields. Since there exist representations of the YB algebra for $q=\infty$, also bosonic QFT may be described in this framework.

The $S=\infty$ representation of the XXX magnet relates to the SU(2) principal chiral σ model (PCM), as was developed in [13]. Let us recall that the physical particle states of this model transform under the $SU(2)_L\times SU(2)_R$ group. The counting of states in the BA equation [13] and our derivation (§ 4, see [14] for details) show that only the $SU(2)_L$ singlet sector of the model is described by the H and P associated through (2.15) to the infinite-S limit of the R matrix [15]

$$R_{12}(\theta)=\frac{\Gamma(2S+1+i\theta)\Gamma(\mathbb{J}+1-i\theta)}{\Gamma(2S+1-i\theta)\Gamma(\mathbb{J}+1+i\theta)}. \tag{2.35}$$

Here the operator $\mathbb{J}$ is defined by

$$\mathbb{J}(\mathbb{J}+1) = 2S(S+1) + 2\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \tag{2.36}$$

where $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are spin-S operators ($\boldsymbol{S}_1^2 = \boldsymbol{S}_2^2 = S(S+1)$) acting on the horizontal and vertical spaces respectively.

In other words, the Hamiltonian of the quantum PCM *does not follow* from the vertex constructions (2.15) and (2.35), even in the scaling limit. Only at the classical level can an equivalence be established between the respective classical analogues [13]. Although (2.15) and (2.36) do not provide at $S = \infty$ the full PCM Hamiltonian, they correctly reproduce its restriction to the SU(2) singlets, and this is sufficent to calculate all particle masses as well as the invariant S-matrix amplitudes. The same considerations apply to the anisotropic SU(2) PC field [16], from which the O(3) non-linear σ model can be obtained.

The lattice current construction, (2.22)-(2.27), also applies to the PCM. For large θ the R matrix (2.36) admits a semiclassical expansion of the type (2.28). Therefore the whole construction holds. It must be noted that we have once again only one conserved and curvatureless matrix current: either the one associated with $SU(2)_R$ or that associated with $SU(2)_L$.

In conclusion, the light-cone transfer matrices U_R and U_L associated with each integrable gapless vertex model yield integrable and massive QFT in the continuum limit. Since the scaling limit can sometimes be performed in several inequivalent ways, one can construct different QFT from a unique given vertex model.

Depending on q, the number of allowed states per link, one finds fermionic theories without internal degrees of freedom ($q = 2$, from the six-vertex model to the MTM), fermionic models with internal symmetries ($q \geqslant 2$) and bosonic models ($q = \infty$).

3. Coordinate Bethe ansatz (CBA)

We derive in this section the CBA for a general class of 2D relativistic chiral-invariant fermionic models (CFM). A CBA also exists [1] for an important non-chiral case: the MTM. The MTM is special in a sense, since there exists a complete light-cone lattice regularisation based on the six-vertex model. The U(1) invariance of this R matrix is identified with bare particle conservation and leads to a proper identification of the canonical lattice fields ψ_n and ψ_n^+. In the general case, the symmetries of the R matrix are not obviously linked to a canonical field structure. This implies that the scaling limit of the light-cone vertex models in general describes only a sector (the zero-chirality sector, to be precise) of a continuum QFT which can be locally written in terms of bare canonical operators. The CBA exactly provides the means for the proper identification of such continuum QFT.

Consider a CFM described by the Lagrangian

$$\mathscr{L} = \mathrm{i}\bar{\psi}\partial\!\!\!/\psi + \psi_{a+}^+ \psi_{b-}^+ V_{ab,cd} \psi_{c+} \psi_{d-} \tag{3.1}$$

where V is a constant Hermitian matrix acting in the internal q-dimensional spaces and determining the interaction between right movers (ψ_+) and left movers (ψ_-). We choose here as Dirac matrices

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \gamma^5 = \gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so that $\gamma_5\psi_\pm = \pm\psi_\pm$. Equation (3.1) coincides with (1.11) upon using $\gamma^\mu \otimes \gamma_\mu = \mathbb{1}\otimes\mathbb{1} - \gamma^5 \otimes \gamma^5$. For generic internal space and generic V, (3.1) essentially describes all relativistic chiral-invariant (and left-right symmetric) fermionic QFT. A further generalisation is possible by breaking the left-right symmetry using different left and right internal spaces. Such a model is considered in [17] in connection with the Wess-Zumino-Witten (WZW) σ model [18].

The Hamiltonian of the model (3.1) is

$$H = \int_{-L/2}^{L/2} \mathrm{d}x(-\mathrm{i}\psi^\dagger\gamma^5\partial_x\psi + \psi^\dagger_{a+}\psi^\dagger_{b-}V_{ab,cd}\psi_{d-}\psi_{c+}) \tag{3.2}$$

and commutes with the $\mathrm{U}(1)\times\mathrm{U}(1)$ charges of the theory

$$\begin{aligned} &Q = N_+ + N_- \qquad Q_5 = N_+ - N_- \\ &N_\pm = \int_{-L/2}^{L/2} \mathrm{d}x\psi^\dagger_{a\pm}\psi_{a\pm} \end{aligned} \tag{3.3}$$

so that $\dot{Q} = \dot{Q}_5 = 0$. While Q requires at least the VEV subtraction to be finite in the physical states, Q_5 does not, since the necessarily unbroken chiral symmetry implies $\langle N_+ - N_-\rangle = 0$ in the vacuum.

The generators of the symmetries exhibited by V also commute with H. For example, V may be invariant under the action of a Lie group G in a given representation ρ. That is, under the change $\psi \to \rho(g)\psi$, $g \in \mathrm{G}$, H will be invariant if

$$[\rho(g)\otimes\rho(g)]^\dagger V[\rho(g)\otimes\rho(g)] = V \qquad \forall g \in \mathrm{G}. \tag{3.4}$$

For instance, the most general V invariant under $\mathrm{SU}(n)$ in the fundamental representation is ($q = n$ here)

$$V_{ab,cd} = g\delta_{ad}\delta_{bc} + (g' - g/n)\delta_{ac}\delta_{bd} \qquad 1 \leqslant a, b, c, d \leqslant n \tag{3.5}$$

where g and g' are arbitrary couplings. This interaction can be written as

$$-\psi^\dagger_{a+}\psi^\dagger_{b-}V_{ab,cd}\psi_{c+}\psi_{d-} = \tfrac{1}{2}g\sum_{\alpha=1}^{n^2-1} J_\mu{}^\alpha J^{\mu\alpha} + \tfrac{1}{2}g' J_\mu J^\mu \tag{3.6}$$

where

$$\begin{aligned} &J_\mu{}^\alpha = \bar{\psi}\gamma_\mu T^\alpha\psi \qquad J_\mu = \bar{\psi}\gamma_\mu\psi \\ &T^\alpha = T^{\alpha\dagger} \qquad \mathrm{Tr}\, T^\alpha = 0 \qquad \mathrm{Tr}\, T^\alpha T^\beta = \tfrac{1}{2}\delta^{\alpha\beta}. \end{aligned} \tag{3.7}$$

J^α_μ and J_μ are, respectively, the 'colour' $\mathrm{SU}(N)$ current and the U(1) current. This model (alternatively called the chiral Gross-Neveu model or $\mathrm{SU}(n)$ Thirring model) was exactly diagonalised by the CBA in [19, 20].

Let us now return to the general model (3.2). The CBA approach to the diagonalisation of H consists of three steps.

(i) The canonical anticommutation rules on the fields ψ and $\psi^\dagger$ are given a Fock representation based on the unphysical reference state $|0\rangle$ killed by all ψ:

$$\psi_{a\pm}(x)|0\rangle = 0 \qquad \forall_{a,x}.$$

(ii) The second-quantised Hamiltonian H is reduced to a differential operator h acting on states with a fixed number of pseudoparticles N_+ and N_- which are both conserved. More explicitly, one sets

$$|F\rangle = \int_{-L/2}^{L/2} \mathrm{d}^N x \sum_{a_1\ldots a_N} F_{a_1\ldots a_N}(x_1\ldots x_N)\prod_{j=1}^{N}\psi^\dagger_{a_j\alpha_j}(x_j)|0\rangle \tag{3.8}$$

where $\alpha_j = \pm (1 \leqslant j \leqslant N \equiv N_+ + N_-)$ define any chirality configuration with prescribed N_+ and N_-. Then

$$H|F\rangle = |hF\rangle \tag{3.9}$$

with

$$h = -\mathrm{i} \sum_{j=1}^{N} \alpha_j \frac{\partial}{\partial x_j} + \sum_{j<l} \tfrac{1}{2}(1 - \alpha_j \alpha_l)\delta(x_j - x_l) V_{jl} \tag{3.10}$$

where V_{jl} stands for the operator acting as V on the tensor product of the jth and lth internal subspaces. It is useful to introduce here the following notation: for any set of quantities $\boldsymbol{q} = (q_1, \ldots, q_N)$ associated with the N pseudoparticles, let $\boldsymbol{q}_+ = (q_{j_1}, \ldots, q_{j_{N_+}})$, $1 \leqslant j_1 < j_2 < \ldots < j_{N_+} \leqslant N$, $\alpha_{j_r} = +$, be the restriction of $\boldsymbol{q}$ to the right-moving pseudoparticles; similarly, one defines $\boldsymbol{q}_-$, which is clearly complementary to $\boldsymbol{q}_+$. The Hilbert space at fixed N_+, N_- is formed by the square-integrable wavefunctions $F_{a_1 \ldots a_N}(x_1 \ldots x_N)$ which are separately totally antisymmetric in the two sets of index pairs $(\boldsymbol{xa})_+$ and $(\boldsymbol{xa})_-$.

(iii) h is diagonalised on CBA states $|\boldsymbol{k}, \Phi\rangle$, whose wavefunctions are given by

$$\langle \boldsymbol{xa} | \boldsymbol{k}, \Phi \rangle = \left[\prod_{j=1}^{N} \exp(\mathrm{i} k_j x_j) \right]_{N_+, N_-} \sum_{Q \in S_N} \theta(x_Q) [S(Q)\Phi]_{a_1 \ldots a_N}. \tag{3.11}$$

In this expression, $[\ldots]_{N_+ N_-}$ means complete antisymmetrisation with respect to $\boldsymbol{x}_+$ and $\boldsymbol{x}_-$ separately; $\boldsymbol{k} = (k_1, \ldots, k_N) = \boldsymbol{k}_+ \cup \boldsymbol{k}_-$ is an arbitrary set of pseudomomenta (with $\boldsymbol{k}_+$ and $\boldsymbol{k}_-$ separately given by all distinct entries); $\Phi = (\Phi_{a_1, \ldots, a_N})$ is any global internal state; the sum $\Sigma_{Q \in S_N}$ runs over all permutations of N objects, which form the symmetric group S_N; $\theta(x_Q) \equiv \theta(x_{Q_1} < \ldots < x_{Q_N})$ is 1 if the argument is true and 0 otherwise; and, finally, the Q-dependent unitary S matrix $S(Q)$ is obtained by taking the product, along *any* path of neighbouring transpositions connecting Q to the identity, of the unitary two-body S matrices:

$$S_{jl} = \begin{cases} P_{jl} & \alpha_j = \alpha_l \qquad (3.12a) \\ \dfrac{\mathrm{i}(\alpha_j - \alpha_l) - V_{jl}}{\mathrm{i}(\alpha_j - \alpha_l) + V_{jl}} & \alpha_j \neq \alpha_l \qquad (3.12b) \end{cases}$$

where P_{jl} is the exchange operator between the jth and lth internal subspaces. $S(Q)$ does not depend on the particular path chosen to reach Q, thanks to the (discrete) factorisability conditions satisfied by S_{jl}:

$$S_{jl} S_{lj} = 1 \tag{3.13a}$$

$$S_{jl} S_{kn} = S_{kn} S_{jl} \qquad j, l, k, n \text{ all distinct} \tag{3.13b}$$

$$S_{jk} S_{jl} S_{kl} = S_{kl} S_{jl} S_{jk}. \tag{3.13c}$$

The eigenvalues of the Hamiltonian H and momentum operator P on $|\boldsymbol{k}, \Phi\rangle$ are the same as those of pure plane waves:

$$H|\boldsymbol{k}, \Phi\rangle = \left(\sum_{j=1}^{N} \alpha_j k_j \right) |\boldsymbol{k}, \Phi\rangle \tag{3.14a}$$

$$P|\boldsymbol{k}, \Phi\rangle = \left(\sum_{j=1}^{N} k_j \right) |\boldsymbol{k}, \Phi\rangle. \tag{3.14b}$$

It is also easy to check that the CBA state $|\boldsymbol{k}, \Phi\rangle$ has the correct symmetry properties, since (3.12*a*) implies

$$S(Q_+Q_-Q) = Q_+Q_-S(Q)$$

where $Q_\pm$ acts only on $\boldsymbol{a}_\pm$, respectively.

While the form of S_{jl} for $\alpha_j = \alpha_l$ is fixed by the requirement that the CBA states have the right discontinuities to cancel the δ functions in h (equation (3.10); we have assumed the convention $\delta(x)\theta(x) = \frac{1}{2}\delta(x)$), the form of S_{jl} for $\alpha_j = \alpha_l$ is dictated by factorisability alone. In a sense, it contradicts the naive expectation that non-interacting equal-chirality pseudoparticles should have a unit S matrix. Actually, any discontinuity in $x_j - x_l$ for $\alpha_j = \alpha_l$ is possible, since the kinetic operator in (3.10) is completely blind with respect to them.

Several comments are in order before we proceed any further. Step (i) might appear as a sacrilege to a relativistic field theorist. The state $|0\rangle$ certainly does not belong to the physical Hilbert space based on the true vacuum state. However, as soon as the original theory (3.2) is given an explicit UV cutoff Λ (which at any rate is always implicit when writing down bare Lagrangians or Hamiltonians), then the state $|0\rangle$ and the true ground state (the physical vacuum at fixed Λ and L) live in the same cutoff Hilbert space. When $\Lambda \to \infty$ they separate in a Λ-dependent way and act as reference states of two orthogonal Fock spaces supporting inequivalent representations of the canonical anticommutation rules. The real problem is how to introduce the UV cutoff Λ without spoiling integrability, which at this stage coincides with the possibility of exactly diagonalising h, the 'first-quantised' operator emerging at step (ii). Indeed, the presence of a UV regularisation must change h by, for example, smearing the δ functions in (3.10), introducing higher derivatives, or replacing them by finite differences, and so on. The crucial point is that step (iii) requires h to have precisely the singular form (3.10). The presence of δ functions in h implies the presence of discontinuities in the CBA wavefunctions (3.11). In turn this means that the CBA states have support extending to infinite momentum in Fourier space. They are not physically cut off. The conventional UV regularisation in the CBA approach is only superficially a true regularisation. It consists of putting a sharp cutoff on the pseudomomenta $k_1, \ldots, k_N$ in order to make the energy eigenvalues (3.14*a*) bounded from below. Clearly, however, this does not correspond to a regularisation of the fields $\psi(x)$ and $\psi^+(x)$ themselves, which still anticommute in a singular way. This very unconventional cutoff procedure can be a source of deep trouble, as we shall see shortly.

For the moment, let us make another observation on step (iii). Its validity does not put any constraint whatsoever on the interacting matrix V. This is because only *discrete* factorisability is required on the two-body S matrices (3.13), and this is guaranteed by the choice (3.12*a*) alone, independent of the particular form of V. Continuous factorisability is required only at a second stage, when the allowed values of the pseudomomenta are explicitly necessary. Let us now turn to this second stage.

This consists of the imposition of periodic boundary conditions on the CBA states. Thanks to the factorisability relations (3.14), this imposition determines the following, finite-dimensional, eigenvalue equations for the state Φ of (3.11):

$$Z_j\Phi = \exp(-ik_jL)\Phi \qquad j = 1, 2, \ldots, N \tag{3.15}$$

where

$$Z_j = S_{jj-1} \ldots S_{j1}S_{jN} \ldots S_{jj+1}. \tag{3.16}$$

The system (3.15) is consistent since $[Z_j, Z_l] = 0$, again thanks to (3.13). Actually, one easily verifies that there are only two distinct Z, since

$$Z_j = Z_{\alpha_j} = Z_{\pm}. \tag{3.17}$$

From (3.14) and (3.15), we now obtain

$$H = \frac{2\pi}{L} \sum_{j=1}^{N} \alpha_j n_j + \frac{\mathrm{i}}{L} (N_+ \log Z_+ - N_- \log Z_-) \tag{3.18a}$$

$$P = \frac{2\pi}{L} \sum_{j=1}^{N} n_j + \frac{\mathrm{i}}{L} (N_+ \log Z_+ + N_- \log Z_-) \tag{3.18b}$$

where the branches of the logarithms are fixed, so that the integers n_j ($n_j \neq n_l$ for $\alpha_j = \alpha_l$) are free BA eigenvector parameters. Since $Z_{\pm}$ also satisfy the identity $Z_+^{N_+} Z_-^{N_-} = 1$, we see that the momentum is properly quantised in multiples of $2\pi/L$.

Even before an explicit solution of the eigenvalue problem (3.15) is available, we see that the CBA has split the energy-momentum eigenvalues into two distinct and independent contributions. The first, given by integers n_j, is exactly the same as a system of free fermions without any internal degree of freedom. The second contribution depends only on the internal state Φ. This is a general feature of the CBA in relativistic models.

Up to this point, no restriction on the interaction matrix V has been imposed. The possibility of completely integrating the discrete eigenvalue problem (3.15) imposes some restrictions now. In particular, we demand that the two-body matrices S_{ij} satisfy a much stronger version of (3.13), which is continuous factorisability. This means that S_{ij} should be the value at $\theta = \frac{1}{2}(\alpha_i - \alpha_j)$ of some unitary matrix $S_{ij}(\theta)$, with θ an arbitrary complex parameter, satisfying the YB algebra

$$S_{jk}(\theta) S_{jl}(\theta + \theta') S_{kl}(\theta') = S_{kl}(\theta') S_{jl}(\theta + \theta') S_{jk}(\theta). \tag{3.19}$$

Of course, for $\theta = \frac{1}{2}(\alpha_j - \alpha_k)$ and $\theta' = \frac{1}{2}(\alpha_k - \alpha_l)$, (3.19) implies (3.13$c$). Equation (3.13$a$) is now replaced by $S_{jk}(\theta) S_{jk}(-\theta^*) = 1$, i.e. $S_{jk}(\theta)^{\dagger} = S_{jk}(-\theta^*)$, while (3.13$b$) needs no generalisation since it is true for any two-body matrix. Notice that (3.12a), $S_{ij} = P_{ij}$ for $\alpha_i = \alpha_j$, implies that $S_{ij}(\theta)$ must be a *regular* solution of the YB algebra:

$$S_{ij}(0) = P_{ij}. \tag{3.20}$$

With the request that V be such that an integrable $S_{jl}(\theta)$ can be found, the problem (3.15) is solved along the usual lines of the quantum inverse scattering method (QISM) [1]. The matrix $S_{jl}(\theta)$ is identified with the local weights $t_{ab}(\theta)$ of § 2

$$S_{jl}(\theta)_{a_1 \cdots a_N}^{a_1 \cdots a_N} = [t_{a_j a_l}(\theta)]_{a_j a_j} \prod_{k \neq j,l} \delta_{a_k}^{a_k} \tag{3.21}$$

so that the formulation (3.19) of the YBA is mapped into the formulation (2.2) (with $R = SP$, P = exchange operator). One then extends the rapidities α_j to arbitrary complex values θ_j (the inhomogeneities of § 2) and explicitly verifies that

$$Z_{\pm} = \tau(\theta_0 = \pm 1; \{\theta_j = (-1)^{j+1}\}) \tag{3.22}$$

where $t(\theta_0; \{\theta_j\})$ is the transfer matrix of (2.1). We see that, by choosing $N_+ = N_-$ and appropriately scaling the spectral parameter θ, one can map $Z_{\pm}$ into the light-cone transfer matrices U_{R} and U_{L}:

$$Z_+ = U_{\mathrm{R}} \qquad Z_- = U_{\mathrm{L}}^{\dagger}. \tag{3.23}$$

This relation can also be derived directly from the definition (3.16) of $Z_\pm$ and that of $U_{R,L}$ of [2] (see figure 4), and holds in any case, even if continuous factorisability (3.19) cannot be established. Indeed, it is sufficient to make a precise choice for the arbitrary chirality configuration $\alpha_1, \ldots, \alpha_N$, i.e. $\alpha_{2j-1} = -\alpha_{2j} = +$, $j = 1, \ldots, N_+$ (recall $N_+ = N_-$), and exploit the regularity condition $S_{ij} = P_{ij}$ when $\alpha_i = \alpha_j$.

Thanks to (3.23) we can identify the energy-momentum operators defined on the light-cone lattice by (2.15) with the chiralless sector (i.e., the n_j-independent sector) of the energy-momentum of the general chiral fermionic model (3.1). Comparing (3.18) (in the case $N_+ = N_-$), (2.15) and (3.23), we need only set $L = N_+ a$, where a is the lattice spacing. Of course, from the point of view of the continuum field theory (3.1), this identification is not quite obvious. It becomes natural only after a cutoff Λ is imposed on the integers n_j by, say,

$$(2\pi/L)\alpha_j n_j \geqslant -\Lambda. \tag{3.24}$$

Then one should minimise H for fixed Λ, determining the cutoff ground state $|\Omega\rangle_\Lambda$. In the integrable cases when this can be explicitly performed, one finds that, in the ground state, $N_+ = N_- = N_0 \sim \Lambda L$ and therefore $\Lambda \sim a^{-1}$ indeed plays the role of a UV cutoff.

In conclusion, from the point of view of complete integrable models, we see that, given a solution of the YB relations (2.2) or (3.19) satisfying the regularity condition, one can either construct a light-cone vertex system characterised by the diagonal-to-diagonal transfer matrices U_R and U_L or a fermionic QFT of type (3.1) having an interaction matrix V related to the solution $S_{jk}(\theta)$ of the YBA by (3.12*b*). In both cases, the BA equations governing the eigenvalue-eigenvector structure of $U_{R,L}$ or $Z_\pm$ are identical. Of course, the QFT model has the extra freedom of independently varying N_+ and N_- away from their ground state value N_0. This, however, does not affect in any way the mass spectrum or the S matrix.

Actually, there is another, more serious, difference between the two above-mentioned procedures. While the light-cone lattice approach is a perfectly well defined means of constructing relativistic, non-trivial 2D QFT in the scaling limit†, the fermionic approach suffers from the cutoff subtleties previously mentioned. In other words, it is not obvious that the CBA solution of the theory (3.1), even in the full integrable case characterised by (3.19), is the correct one, lying in the same universality class, at infinite UV cutoff, of conventional perturbation theory and other more traditional field-theoretic methods. From a purely technical point of view, let us observe that the CBA state completeness, in the field-theoretic sense, is very difficult to ascertain (in contrast, completeness of BA states for fixed $N_\pm$ is well understood, just as in spin systems [1]). This is because the cutoff procedure (3.24) breaks completeness in each fixed (N_+, N_-) sector, and when Λ is removed to ∞, also N_+, N_- tend to ∞ in any physical state. From another point of view, let us observe that the chiral currents

$$J_{R,L}(x) \equiv J_\pm(x) = \psi^\dagger_{a\pm}\psi_{a\pm} \tag{3.25}$$

are replaced by multiplication operators

$$J_\pm(x) \to \sum_{j=1}^{N_\pm} \delta(x - x_{j\pm}) \qquad x_{j\pm} \in \boldsymbol{x}_\pm \tag{3.26}$$

† Strictly speaking, there is also here a price to be paid: up to now, a canonical lattice field structure has not been generally available.

in the CBA Fock representation (3.6) based on the *unphysical* vacuum $|0\rangle$. Hence the commutators $[J_\pm(x), J_\pm(y)]$ vanish identically for any $N_\pm$ and will never give rise to a Schwinger term, even when $\Lambda \to \infty$. However, physical chiral currents, as is well known, must have anomalous commutators, by Schwinger's theorem [21]. The only way out is that the canonical $J_\pm$ of (3.25) are not the physical $U(1) \times U(1)$ currents; not just because they are not VEV subtracted (a c-number subtraction will not change the commutator), but because the fields ψ^+ and ψ are evaluated at the same point *before* the UV cutoff is sent to ∞. In a theory like the CGN of (3.6), where the interaction is dominated by the $SU(n)$ current-current coupling (we recall that the trajectory $g' = 0$ is stable under renormalisation up to the second loop), this problem with the $U(1) \times U(1)$ currents is harmless. Indeed, the $SU(n)$ currents are obviously not mapped into multiplication operators by the CBA (although their commutator is still free of central terms), and to get the described Schwinger terms in the commutator of physical $U(1) \times U(1)$ currents, it is presumably sufficient to split $\psi^+(x)\psi(x) \to \psi^+(x+\varepsilon)\psi(x)$ and take $\varepsilon \to 0$ only after the UV cutoff Λ is sent to ∞ in the CBA construction.

Quite different is the situation in the massless Thirring model (the $g = 0$ case of (3.6)). Here the CBA is bound to fail. By mapping the $U(1) \times U(1)$ currents (which alone enter the interaction) into diagonal operators (equation (3.26)), the CBA implies a complete decoupling of physical excitations. As the exact solution shows, both in operator [22] and functional language [23], this decoupling is actually prevented only by the axial anomaly, which is just a consequence of Schwinger terms (the crucial role of Schwinger terms could already be seen in the old fashioned non-covariant perturbation theory). For a correct BA lattice construction of the massless Thirring model see, e.g., [24].

In the case of chiral fermionic models with any non-Abelian simple symmetry group in the fundamental representation, it is possible to perform a non-trivial check that the CBA solution should indeed lie in the same universality class of standard perturbative approaches. For all these models, the S matrices $S_{jl}(\theta)$ satisfying the YB equations (3.19) are known [3, 4]. They all enjoy the semiclassical expansion (1.4) for large θ (recall that the S and R matrices are connected by the exchange operator $P: S = PR$). Thus

$$S(\theta) \simeq 1 + (1/\mathrm{i}\theta)(\Pi + \lambda) + O(1/\theta^2). \tag{3.27}$$

Due to (3.12b), the large-θ form of the interaction matrix V is

$$V(\theta) \simeq (1/\theta)(\Pi + \lambda) + O(1/\theta^2) = (1/\theta)\left(\sum_\alpha t^\alpha \otimes t^\alpha + \lambda\right) + O(1/\theta^2)$$

and, with the natural identification $\theta = 1/g$ (compare this relation with the 'bare' scaling limit of § 2), the Lagrangian (3.1) becomes

$$\mathscr{L} = \mathrm{i}\bar{\psi}\partial\!\!\!/\psi - \tfrac{1}{2}g\left(\sum_\alpha J_\mu{}^\alpha J^{\mu\alpha} + \lambda J_\mu J^\mu\right) + O(g^2) \tag{3.28}$$

i.e, the typical G-invariant current-current interacting theory, up to $O(g^2)$ (for special cases, like $G = SU(n)$, it is possible to obtain the form (3.28) without higher-order corrections, by choosing $\theta = \theta(g) = 1/g + \alpha_0/g^2 + \ldots$ and an overall phase factor for $S_{jl}(\theta)$, see § 4).

We can now study the model (3.28) in conventional perturbation theory, up to first loop. The $O(g^2)$ terms are then irrelevant as far as divergences are concerned. Calculations are simplified by introducing auxiliary, non-propagating fields A_μ^α, to split each

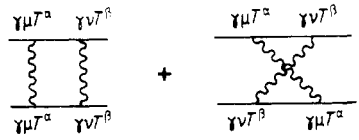


Figure 6. The two logarithmically divergent ladder diagrams. The wavy lines represent the contraction of two auxiliary fields A_μ^α.

four-fermion vertex into two $A_\mu^\alpha J^{\mu\alpha}$ vertices. Due to the two-dimensional property $\gamma_\mu\gamma_\nu\gamma^\mu = 0$, one can easily realise that only the two ladder diagrams of figure 6 are logarithmically divergent. The coefficient of this divergence is given by

$$-\frac{g^2}{2\pi}\sum_{\alpha,\beta}[t^\alpha, t^\beta]\otimes[t^\alpha, t^\beta] = g^2\frac{C_2}{2\pi}\sum_\alpha t^\alpha\otimes t^\alpha \tag{3.29}$$

where C_2 is the quadratic Casimir operator of the adjoint representation. Notice that the Abelian $J_\mu J^\mu$ coupling cannot contribute to this divergence exactly because of its Abelian nature.

The result (3.29) demonstrates the asymptotic freedom of the model and provides the value $-(C_2/2\pi)$ for the first coefficient of the β function. On the other hand, this same coefficient can be calculated from the scaling form of the dynamically generated mass in the BA construction, (2.18). It equals κ^{-1}. A look at table 1 then confirms that $\kappa = 2\pi/C_2$ for all simple Lie algebras (we have chosen the overall normalisation of the Lie algebra such that the shortest root has unit length). The very different origin of the same numbers κ provides a rather non-trivial check of the validity of the BA for the CFM in the fundamental representations.

4. The CBA for multiflavour models and the principal chiral σ model

In the preceding section we reviewed the CBA, discussed its main shortcomings, and presented an Abelian example in which it fails (the MTM), as well as a large class of non-Abelian current-current models (equation (3.28): the generalisation to any simple Lie group of the SU(n) CGN model), in which it is successful. Indeed, the CBA for this class of models passes other checks besides the one-loop universality check examined here; for instance, it agrees with the $1/N$ expansion and with semiclassical methods.

In this section, we shall present another very important example, in which the CBA of § 3 fails. This is the multiflavour generalisation of model (3.28), when the 'colour' currents of the Lie group G are

$$J_\mu^\alpha = \sum_{r=1}^{N_f}\bar\psi_r\gamma_\mu t^\alpha\psi_r. \tag{4.1}$$

This class of models is important also because of its connection with the PC σ model on the group G [25, 26]. To be specific, we shall consider only the simplest case $G = \mathrm{SU}(n)$, i.e. the multiflavour chiral Gross-Neveu (MCGN) model. In this case, the internal space has total dimensionality $q = nN_f$.

First let us see how the CBA works for the MCGN. The flavour acts solely as a spectator of the colour interaction (though its presence has non-trivial consequences through the Pauli exclusion principle); hence the colour two-body S matrices are

unchanged. In the continuous form satisfying (3.19) they are, for $g' = 0$ (compare with (3.20)),

$$S_{jl}(\theta) = \frac{2ni\tilde{g}\theta + (n-1)g}{2ni\tilde{g}\theta - (n-1)g} \frac{1 + i\theta P_{jl}}{1 + i\theta} P_{jl} \tag{4.2}$$

where

$$\tilde{g} = g\left(1 - \frac{n^2-1}{4n^2} g^2\right)^{-1}$$

and the discrete S_{ij} of (3.12*a*) and (3.12*b*) are recovered when

$$\theta \to (1/2\tilde{g})(\alpha_j - \alpha_l). \tag{4.3}$$

To take flavour into account in an integrable way there are two possibilities. The most natural possibility consists of taking a unit S matrix in flavour space; it will be analysed first. In this case, the CBA wavefunctions (equation (3.11)) are modified into

$$\langle \boldsymbol{xar} | \boldsymbol{k}, \Phi, \Psi \rangle = \left[\Psi_{\boldsymbol{r}} \prod_{j=1}^{N} \exp(i\boldsymbol{k}, x_j) \right]_{N_+ N_-} \sum_{Q \in S_N} \theta(x_Q)[S(Q)\Phi]_a \tag{4.4}$$

where $\Psi = (\Psi_{r_1, \ldots, r_N})$ is the flavour state. $[\ldots]_{N_+ N_-}$ now denotes complete antisymmetrisation in the pair sets $(\boldsymbol{xr})_+$ and $(\boldsymbol{xr})_-$. Hence the pseudomomenta $\boldsymbol{k}_+$ and $\boldsymbol{k}_-$ must obey an exclusion principle of order N_f, i.e. the same entry can appear in $\boldsymbol{k}_\pm$ at most N_f times. Although colour and flavour are decoupled in the CBA states (3.30), the global internal state being just a direct product $\Phi_{a,r} = \Phi_a \Psi_r$, it should be clear that one can always find a complete basis in each (N_+, N_-) sector of the decoupled form (3.30). As we previously observed, however, 'first-quantisation' completeness in each (N_+, N_-) sector is not enough; one needs a (much more difficult even to define) field-theoretic, i.e. 'second-quantisation', completeness. To clarify this question, suppose we proceed for (3.30) as in the general case. After periodic boundary conditions are imposed we would arrive at (3.18*a*) and (3.18*b*), which express the energy-momentum eigenvalues in terms of arbitrary integers $\boldsymbol{n} = (n_1, \ldots, n_N)$ (now satisfying an exclusion principle of order N_f), and the eigenvalues $z_\pm$ of the $Z_\pm$ operators (equations (3.16) and (3.17)). For fixed $N_\pm$, the number $z_\pm$ has a lower bound and so does $\log z_\pm$, since the branch is fixed. The unboundedness from below of the relativistic Hamiltonian h is due to the integers $\boldsymbol{n}$, which can be arbitrarily large. The standard BA procedure is to cut them off as in (3.24). It thus becomes possible to find the ground state, for fixed cutoff, and excited (physical) particle states. In this way, when the cutoff Λ on $\boldsymbol{n}$ is sent to ∞, one reconstructs the physical Fock space.

Is this the right approach, in the sense that all physical states can be obtained in this way? For the MCGN, which has $N_f > 1$, the answer is 'no' (it is 'yes' only for $N_f = 1$, i.e. for the standard CGN). There are basically two independent arguments to see this. First, in the ground state and excited states, colour and flavour are fully decoupled for any value of cutoffs Λ and L; hence colour and flavour are decoupled in the 'physical' Fock space. This is due to the peculiar form of the CBA states (4.4): they achieve complete antisymmetry in $(\boldsymbol{xar})_\pm$ by multiplying a factor completely antisymmetric in $(\boldsymbol{xr})_\pm$, $[\Psi_{\boldsymbol{r}} \exp(i\boldsymbol{k}x)]_{N_+, N_-}$, and a factor completely symmetric in $(\boldsymbol{xa})_\pm$, $\Sigma_Q \theta(x_Q)(S(Q)\Phi)_a$. However, this colour-flavour decoupling in the physical sector contradicts simple perturbation theory. In case one starts invoking non-perturbative effects, there is the second argument, which shows that this decoupling is not due

to the interaction but, rather, is a CBA pathology. Consider the non-interacting case, $g=0$ and, since in this case the $(+)$ and $(-)$ worlds do not see each other, let us focus our attention on, say, the $(+)$ pseudoparticles. The CBA states (4.4) apply equally well to the $g=0$ case, and lead to a bookkeeping of particle states very different from the conventional bookkeeping based on a Fermi sea filled by nN_f negative energy particles per level. Recall that, by (3.12), S_{jl} at $V=0$ is 1 for $\alpha_j \neq \alpha_l$, but is $P_{jl} \neq 1$ for $\alpha_j = \alpha_l$. The BA bookkeeping is based on the energy formula (equation (3.18a)):

$$E_+ = \frac{2\pi}{L} \sum_{j=1}^{N_+} n_j + \mathrm{i} \frac{N_+}{L} \log Z_+ \tag{4.5}$$

where Z_+ turns out to be the cyclic operator on colour indices

$$(Z_+ \Phi)_{a_1 \ldots a_{N_+}} = \Phi_{a_2 \ldots a_{N_+} a_1} \tag{4.6}$$

as is evident from (3.16) and $S_{jl} = P_{jl}$. Since $Z_+^{N_+} = 1$, E_+ is properly quantised in multiples of $2\pi/L$. The eigenvectors of Z_+ are very well known (see, for example, the review in [27] for a very clear and simple treatment), hence given a cutoff Λ such that $2\pi n_j > \Lambda L$, the ground state can be determined and all the degeneracies of excited levels can be calculated (the strength of the BA approach lies in the fact that one can do the same in the interacting case, with essentially no extra complications). In such computations, one must 'fold' the degeneracies of the first term in (4.5), which are proper to a Fermi system with N_f internal degrees of freedom, with those of the second term, which follow the rather complicated rules of the BA equations. In [27] the calculations are performed *in the case* $N_f = 1$ (and $n=2$), and complete agreement is found with the conventional bookkeeping. (This is one of the most important reasons why the answer for the one-flavour CGN is 'yes'; the BA construction is complete.) For this very reason, however, no agreement is possible in the $N_f > 1$ case; i.e., the degeneracy of each energy level, $2\pi n/L$, is much larger in the conventional bookkeeping than in that based on (4.5). Indeed, the contribution of the colour part, $(\mathrm{i}N_+/L)\log Z_+$, is the same regardless of N_f and of the way the n_j levels are filled; in other words, colour and flavour are completely decoupled in contrast with conventional bookkeeping. The latter is really based on the group $\mathrm{U}(nN_f)$, whose irreducible representations (irreps) break into irreps of $\mathrm{U}(N_f) \times \mathrm{SU}(n)$ which are *no longer degenerate* under the energy formula (4.5). The fermion packing is much tighter in the standard approach, so that the density of states is larger. The BA treatment of free chiral fermions with separated colour and flavour (i.e. a $\mathrm{U}(N_f) \times \mathrm{SU}(n)$ bookkeeping) misses a large number of states; this happens although the CBA states (4.4) are complete in each N_+ pseudoparticle sector at infinite Λ, i.e. when the integers n_j are unconstrained. When the interaction between left and right movers is turned on, the symmetry is indeed reduced from $\mathrm{U}(nN_f) \times \mathrm{U}(nN_f)$ down to $\mathrm{U}(N_f) \times \mathrm{U}(N_f) \times \mathrm{SU}(n)$, and part of the free-system degeneracies are lifted. Nevertheless, this cannot make complete at $g>0$ a vector basis which was incomplete at $g=0$. This is why the results of [28], where the CBA (4.4) is assumed, are necessarily wrong (besides being, by the way, in contradiction with general field-theoretic expectations).

We mentioned beforehand that there exists another way to take flavour into account while preserving integrability. The analysis of the free chiral fermions performed above suggests that the two-body S matrices should not be trivial in flavour space. Rather, one should take CBA states (3.11) with

$$S_{jl} = S_{jl}^{(\mathrm{colour})} \otimes S_{jl}^{(\mathrm{flavour})} \qquad \Phi = (\Phi_{ar}) \tag{4.7}$$

where $S_{jl}^{(\text{colour})}$, given by (4.2) and (4.3), acts only in colour space, while

$$S_{jl}^{(\text{flavour})} = \begin{cases} P_{jl}^{(\text{flavour})} & \alpha_j = \alpha_l \\ \mathbb{1} & \alpha_j \neq \alpha_l. \end{cases} \tag{4.8}$$

Equations (3.33) and (3.34) imply that S_{jl} is the total exchange operator P_{jl}

$$P_{jl} = P_{jl}^{(\text{colour})} \otimes P_{jl}^{(\text{flavour})} \tag{4.9}$$

when $\alpha_j = \alpha_l$. Now the free system can be properly described by the CBA; the energy formula (4.5) is modified into

$$E_+ = \frac{2\pi}{L} \sum_{j=1}^{N_+} n_j + \mathrm{i} \frac{N_+}{L} \log Z_+^{(\text{colour})} Z_+^{(\text{flavour})} \tag{4.10}$$

with the integers n_j all distinct. The only subtlety is in the way $Z_+ \equiv Z_+^{(\text{colour})} Z_+^{(\text{flavour})}$ is diagonalised: we can use an $\mathrm{SU}(nN_\mathrm{f})$ bookkeeping, as well as an $\mathrm{SU}(n) \times \mathrm{SU}(N_f)$ one. The former corresponds, in the language of the QISM, to the identification

$$Z_+ = \tau_+(\theta_0 = 0; \{\theta_j = 0\}) \tag{4.11}$$

where $\tau_+(\theta_0; \{\theta_j\})$ is the transfer matrix for right movers constructed according to (2.1) and (3.21) from the 'continuous' S matrices

$$S_{jl}(\theta) = (\mathrm{i}\theta + P_{jl})/\mathrm{i}\theta + 1) \tag{4.12}$$

The $\mathrm{SU}(n) \times \mathrm{SU}(N_\mathrm{f})$ bookkeeping corresponds instead to the choice

$$Z_+ = \tau_+^{(\text{colour})}(0; \{0\}) \otimes \tau_+^{(\text{flavour})}(0; \{0\}) \tag{4.13a}$$

$$S_{jl}(\theta) = \frac{\mathrm{i}\theta + P_{jl}^{(\text{colour})}}{\mathrm{i}\theta + 1} \otimes \frac{\mathrm{i}\theta + P_{jl}^{(\text{flavour})}}{\mathrm{i}\theta + 1} \tag{4.13b}$$

and implies a colour–flavour factorisation: $\Phi_{a_+, r_+} = \Phi_{a_+} \psi_{r_+}$. The $\mathrm{SU}(nN_\mathrm{f})$ bookkeeping leads to the correct degeneracies of the free system, i.e. the BA $\mathrm{U}(1) \times \mathrm{SU}(nN_\mathrm{f})$ bookkeeping is equivalent to the standard $\mathrm{U}(nN_\mathrm{f})$ bookkeeping, as mentioned above. The $\mathrm{SU}(n) \times \mathrm{SU}(N_\mathrm{f})$ approach, on the other hand, is equivalent to the $\mathrm{U}(N_\mathrm{f}) \times \mathrm{SU}(n)$ bookkeeping based on (4.5): in other words, it is wrong.

While for the free system we can choose the right bookkeeping, for the interacting case the dangerous $\mathrm{U}(N_\mathrm{f}) \times \mathrm{U}(N_\mathrm{f}) \times \mathrm{SU}(n)$ bookkeeping is forced upon us by the symmetry reduction. It corresponds to the choice

$$\Phi_{ar} = \Phi_a \Psi_{r_+}^{(+)} \Psi_{r_-}^{(-)} \tag{4.14a}$$

$$Z_\pm = \tau^{(\text{colour})}(\theta_0 = \pm \tilde{g}^{-1}; \{\theta_j = (-1)^{j+1} \tilde{g}^{-1}\}) \otimes \tau_\pm^{(\text{flavour})}(\theta_0 = 0, \{\theta_j = 0\}) \tag{4.14b}$$

where $\tau^{(\text{colour})}(\theta_0; \{\theta_j\})$ is constructed with $S_{jl}^{(\text{colour})}(\theta)$ (equation (4.2); observe also that θ is scaled by $\tilde{g}$ with respect to the general discussion of (3.19)–(3.22)) and $\tau_\pm^{(\text{flavour})}$ is constructed with $(\mathrm{i}\theta + P_{jl}^{(\text{flavour})})(\mathrm{i}\theta + 1)^{-1}$. Quite naturally, this faulty approach leads to BA equations with no coupling whatsoever among colour, flavour and chiral labels n_j. At present, it seems very hard to find a successful cutoff procedure, more sophisticated than the simple bound (3.24) on the n_j, while retaining the singular CBA form (3.11). It is very likely, after all, that the source of trouble is just in the unregularised nature of the first-quantised Hamiltonian h.

To circumvent all these difficulties, it has been proposed [17, 25, 26] that the MCGN model (as well as any other multiflavour model symmetric under a classical Lie group

in the fundamental representation) is actually equivalent to one-flavour CFM of the general type (3.1), with the interaction matrix V taken to be

$$V = 2\mathrm{i}\,\frac{S_T(\theta)-1}{S_T(\theta)+1}\bigg|_{\theta=1/g} \tag{4.15}$$

(i.e. the standard relation (3.12b) for $\alpha_j = \alpha_l = +$), where S_T is a two-body S matrix for the scattering of symmetric tensors of rank N_f. In other words, the idea is to replace the internal space of a multiflavour model, which is obviously reducible under the symmetry group G, by an irreducible internal space corresponding to the highest irrep contained in the original reducible space. The tensor S matrix in (4.15) is not quite arbitrary; it is obtained by 'fusion' [15, 29] from the S matrix for 'particles' in the fundamental representation. As such, it is a regular solution of the YB equations (3.19). Let us briefly illustrate the fusion procedure for the case $G = SU(n)$. Consider the factorised scattering of N_f $SU(n)$ vectors upon another N_f such vectors and assign to the pth vector, $1 \leqslant p \leqslant N_f$, a complex-valued rapidity θ_p. The total scattering matrix is a suitable product of matrices like (4.12), each evaluated at argument $\theta = \theta_p - \theta_{p'}$, if the corresponding two-body scattering involves the vectors p and p'. The many different possible product structures are all equivalent, thanks to the YB algebra (3.19). For special values of the rapidities θ_j (the so-called 'string values'), the full S matrix is projected upon the channel in which two symmetric tensors of rank N_f scatter upon each other. This provides the desired two-body tensor S matrix $S_T(\theta)$ of (4.15). It is the irreducibility under the symmetry group ($SU(n)$ in the case at hand) of the internal space of a CFM which ensures that the BA bookkeeping of excited states is correct.

The physically intuitive argument supporting the above-mentioned substitution is the following (no explicit proof exists in the published literature). The factorisable interaction between $SU(n)$ vectors in the MCGN is antiferromagnetic, being most attractive in the singlet channel. The ground state of the MCGN must contain an infinity of these vector-like pseudoparticles and, due to the extra flavour degeneracy, up to N_f equal chirality pseudoparticles can be located at the same point in space with 'parallel' colours, i.e. in the symmetric tensor representation of rank N_f. The singlet channel of the fused tensor S matrix is the most attractive, leading to the largest phase shift among all singlet channels of the scattering of N_f vectors upon other N_f vectors. Hence the vector pseudoparticles in all the physical states near the true vacuum will tend to form bound states which are symmetric tensors of rank N_f. Within the standard cutoff procedure (3.24), the CBA states (4.4) or the equivalent ones constructed according to (4.14) (with decoupled colour and flavour) certainly do not allow such a bound-state formation, since the pseudoparticle rapidities are 'frozen' to the discrete values $\alpha_j = \pm$. On the other hand, the model with the tensor interaction (4.15) describes only that sector of the MCGN where the bound-state formation is maximum; from this point of view, it can hardly be 'equivalent' to the full MCGN in the field-theoretic sense (see also [30] on this point). Nevertheless, the coloured, massive and chiralless sectors of the two theories are probably equivalent†, at least in the $N_f \to \infty$ limit relevant for the connection to the PC σ model pointed out in the pioneering works [25] and extensively used in [26] also for Lie groups other than $SU(n)$. Therefore, given that the $SU(n)$

† A similar situation is encountered in the BA solution of the multichannel Kondo model [31]: there a suitable regularisation is found that yields a complete tensor bound-state formation, leading to the correct and full description of the electron–impurity magnetic interaction; the electron gas itself, on the other hand, including its charge degrees of freedom, is only partially accounted for.

PC σ model is obtained from the $N_f \to \infty$ limit of the MCGN model; that the massive sector of the latter can be constructed by solving the subsidiary irreducible tensor CF model; and that the PC σ model exhibits dynamical mass generation just like the MCGN but without any massless sector, one could claim, as in [26], to have solved the PC σ model via CBA.

We close this rather lengthy section with a careful investigation of the link between the MCGN model and the PC σ model. We shall show that the $N_f \to \infty$ limit of the MCGN does not reproduce the entire Hilbert space of the PC σ model, but only a subspace of it. Details of this derivation can be found in [14].

Consider the action functionals of the SU(n) PC σ model,

$$\mathscr{A} = \frac{1}{g} \int d^2x \, \mathrm{Tr}(\partial_\mu U)(\partial^\mu U^{-1}) \qquad U \in \mathrm{SU}(n) \tag{4.16}$$

and that of the MCGN model,

$$\mathscr{A}_{N_f} = \int d^2x (i\psi \not{\partial} \psi - \tfrac{1}{2} g J_\mu{}^\alpha J^{\mu\alpha}) \tag{4.17}$$

where

$$J_\mu{}^\alpha = \sum_{j=1}^{N_f} \bar{\psi}_j \gamma_\mu T^\alpha \psi_j \qquad \mathrm{Tr}\, T^\alpha T^\beta = \tfrac{1}{2}\delta^{\alpha\beta}.$$

Using a functional integral argument, Polyakov and Wiegmann claimed [25] that $\mathscr{A}_\infty = \mathscr{A}$ (in the sense, of course, that expectation values calculated with $\mathscr{A}_{N_f}$ tend to those calculated with $\mathscr{A}$ as $N_f \to \infty$). Now consider the partition function $Z_{N_f}(\beta)$ at inverse temperature β for model (4.17). It can be written

$$Z_{N_f}(\beta) = \int D\psi D\bar{\psi} DA \exp\left(\int_0^\beta dx_2 \int dx_1 [i\bar{\psi} \not{D} \psi - (1/g)\, \mathrm{Tr}\, A_\mu A_\mu] \right) \tag{4.18}$$

where $D_\mu = \partial_\mu - iA_\mu$, $A_\mu = A_\mu^\alpha T^\alpha$ and the spacetime signature is Euclidean. The boundary conditions in the Euclidean time x_2 are periodic for A_μ and antiperiodic for ψ and $\bar{\psi}$, as is appropriate for finite-temperature QFT.

Integration over A_μ gives back the action (4.17). Integration over ψ and $\bar{\psi}$ gives the effective action (in gauge-invariant regularisations)

$$\Gamma[A] = \frac{1}{g} \int d^2x \, \mathrm{Tr}\, A_\mu A_\mu - N_f \Gamma_{\mathrm{WZW}}[MM^+] \tag{4.19}$$

where M is a SL($n, \mathbb{C}$)-valued field related to A_μ by

$$A \equiv A_1 - iA_2 = iM^{-1}\partial M \qquad \partial \equiv \partial_1 - i\partial_2 \tag{4.20}$$

and $\Gamma_{\mathrm{WZW}}[\Omega]$, for any invertible Ω, is the (Euclidean) WZW functional [18, 23]. The argument now is that for $N_f \to \infty$, only 'pure gauge' fields $A_\mu = iU^{-1}\partial_\mu U$, corresponding to $M = U \in \mathrm{SU}(n)$, will contribute to the residual functional integration over A_μ, since they make $\Gamma_{\mathrm{WZW}}[MM^+]$ vanish. All other configurations give a strictly negative Euclidean Γ_{WZW} and will be suppressed in the $N_f \to \infty$ limit. The first term in (4.19), quadratic in A_μ, then gives precisely the PC σ model action (4.16).

The missing point in [25] is that the change of variables from A to U allows for a *twisted* boundary condition (BC) on U. To see this properly it is convenient to change variables from A to $M \in \mathrm{SL}(n, \mathbb{C})$ at fixed N_f. Then the periodic BC on A

$$A(x_1, \beta) = A(x_1, 0) \tag{4.21}$$

allows for a *twisted* BC on M

$$M(x_1, \beta) = L_\beta M(x_1, 0) \tag{4.22}$$

where L_β is an arbitrary constant matrix of $SL(n, \mathbb{C})$. Moreover, it is necessary to integrate over all possible twists L_β in order to reproduce in full the A integration. Also the Jacobian of this change of variables should be properly taken into account [14]. The net result is that, when $N_f \to \infty$ and $M \to U \in SU(n)$, also $L_\beta \to U_\beta \in SU(n)$ and one is left with

$$\lim_{N_f \to \infty} Z_{N_f}(\beta) = \int dU_\beta \int_{(U_\beta)} DU \exp(-\mathscr{A}[U]) \tag{4.23}$$

where $\mathscr{A}[U]$ is given by (4.16) and the subscript (U_β) indicates that U satisfies the twisted BC $U(x_1, \beta) = U_\beta U(x_1, 0)$. We now recognise in (4.23) not just the partition function of the PC σ model, but rather the trace [14]

$$\lim_{N_f \to \infty} Z_{N_f}(\beta) = \mathrm{Tr}(\exp(-\beta H) P_0) \tag{4.24}$$

where H is the Hamiltonian of the PC σ model and P_0 is the projector onto singlet states of $SU(n)_L$. It is clear that an analogous expression can be obtained with the projector onto the $SU(n)_R$ singlet, by just setting $A = i(\partial M)M^{-1}$ rather than $A = iM^{-1}\partial M$ as in (4.20). What one cannot obtain are the physical states which transform non-trivially under both $SU(n)_L$ and $SU(n)_R$.

Our result (4.24) explains why the careful counting of BA states in [13] did not reproduce all the states of the SU(2) PC σ model, but only the singlets under either $SU(2)_R$ or $SU(2)_L$. In [13], the setting is essentially that of a light-cone vertex model; the R matrix is the $S \to \infty$ limit of the R matrix (2.36) for symmetric tensors of rank $2S$ under SU(2) (i.e. states of spin S). We have seen in § 3 that such a vertex model is equivalent, in the scaling limit, to the massive sector of a CFM with spin-S internal space. In turn, this should be equivalent to the massive sector of the multiflavour SU(2) CGN with $N_f = 2S$. But as $N_f \to \infty$ this fails to reproduce the entire Hilbert space, giving only the $SU(2)_L$ singlets.

In the SU(2) case it is possible to reconstruct the entire spectrum and S matrix from the knowledge of their restriction to the $SU(2)_L$ singlets. For a generic classical group G, or even exceptional Lie groups, this remains to be fully established. The situation appears even less convincing if Green functions are to be considered. The construction of an explicit, regularised version of the full PC σ model, with a regularised Hamiltonian still exactly diagonalisable, remains an open challenge.

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